

# Superfield Lax formalism of supersymmetric sigma model on symmetric spaces

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**Abstract.** We present a superfield Lax formalism of the superspace sigma model based on the target space  $\mathcal{G}/\mathcal{H}$  and show that a one-parameter family of flat superfield connections exists if the target space  $\mathcal{G}/\mathcal{H}$  is a symmetric space. The formalism has been related to the existence of an infinite family of local and non-local superfield conserved quantities. A few examples have been given to illustrate the results.

## 1 Introduction

In recent years, some investigations have been made to study the infinite number of conserved quantities of symmetric space sigma models, their Poisson bracket algebra and the quantum conservation of these quantities [1–3]. The supersymmetric extension of sigma models based on symmetric spaces  $\mathcal{G}/\mathcal{H}$  have also been studied quite recently, and it has been shown that there exist two classes of local conservation laws; one class of conservation laws corresponds to cohomology of the target manifold and the second class corresponds to higher-spin generalizations of the energy momentum tensor [4]. The investigation of these integrable models has applications in recent advances in superstring theories on AdS backgrounds [5–13]. The model studies in these investigations are in many ways related to the symmetric space sigma model. In all these studies, a formal analysis of the integrability of supersymmetric sigma models on a symmetric space has not been carried out so far, which is what we intend to present in this work.

In this paper we will extend certain results related to the integrability of the supersymmetric sigma models based on  $\mathcal{G}/\mathcal{H}$ . We generalize earlier results of Eichenherr and Forger [14] and show that the supersymmetric sigma model with target space a homogeneous space  $\mathcal{G}/\mathcal{H}$  admits a one-parameter family of flat superfield currents if  $\mathcal{G}/\mathcal{H}$  is a symmetric space. Our main result is to find a superfield Lax formalism in terms of a one-parameter family of flat superfield currents of the supersymmetric sigma model on a symmetric space and relate it to the infinitely many local and non-local superfield conserved quantities of the model. We illustrate our results by giving some explicit examples of sigma models on complex Grassmannians  $U(m+n)/U(m) \times U(n)$  and principal chi-

ral models for which the corresponding symmetric spaces are  $\mathcal{G} \times \mathcal{G}/\mathcal{G}$ .

This paper is organized as follows. In Sect. 2, we give a general theory of the  $\mathcal{G}/\mathcal{H}$  sigma model in superspace. Section 3 contains the Lax formalism of the model in terms of a one-parameter family of flat superfield connections, and it has been shown that a Lax formalism is admissible when  $\mathcal{G}/\mathcal{H}$  is a symmetric space. In Sect. 4, we investigate the existence of local and non-local conserved quantities of the model and compare our results with the results obtained earlier. Section 5 gives an explanation of our investigations with some explicit examples. Section 6 contains our concluding remarks.

## 2 The $\mathcal{G}/\mathcal{H}$ sigma model in superspace

We begin by defining a compact symmetric space and formulate the supersymmetric non-linear sigma model on a symmetric space  $\mathcal{G}/\mathcal{H}$ . For the general structure of the model, we follow the treatment of the model adopted for the bosonic model [14–17] and the supersymmetric model [4].

A symmetric space is defined as follows. Let  $\mathcal{G}$  be a compact Lie group with Lie algebra  $\mathfrak{g}$  and let  $\mathcal{H}$  be its subgroup with a Lie algebra  $\mathfrak{h}$ . Let  $\sigma$  be a linear automorphism  $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$  such that  $\sigma^2 = 1$ . This means that  $\sigma$  has eigenvalues  $\pm 1$  and it splits the algebra  $\mathfrak{g}$  into orthogonal eigenspaces corresponding to these eigenvalues. This automorphism is called an involutive automorphism. This causes the canonical decomposition of  $\mathfrak{g}$  as follows:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{k}, \quad (1)$$

where  $\mathfrak{h}$  and  $\mathfrak{k}$  are the (+1) and (−1) eigenspaces of  $\sigma$  ( $Z_2$  grading of  $\mathfrak{g}$ ) and the action of  $\sigma$  on the vectors of  $\mathfrak{g}$ ,  $\mathfrak{h}$  and

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$\mathbf{k}$  is as follows:

$$\begin{aligned} [\sigma(X), \sigma(Y)] &= \sigma[X, Y] \text{ for } X, Y \in \mathfrak{g}, \\ \sigma(X) &= X \text{ for } X \in \mathfrak{h}, \\ \sigma(X) &= -X \text{ for } X \in \mathfrak{k}. \end{aligned}$$

It is clear from the above that  $\mathfrak{h}$  is a subalgebra but  $\mathfrak{k}$  is not. In fact,  $\mathfrak{k}$  is the orthogonal complement of  $\mathfrak{h}$  in  $\mathfrak{g}$ . The following Lie brackets hold:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}. \quad (2)$$

The algebra  $\mathfrak{h}$  satisfying the relation (2) is called a symmetric space subalgebra. The coset homogeneous space  $\mathcal{G}/\mathcal{H}$  with involutive automorphism  $\sigma$  and admitting a canonical decomposition (1) obeying (2) is called a symmetric space [18, 19].

In order to formulate a supersymmetric sigma model on  $\mathcal{G}/\mathcal{H}$  in  $(1+1)$ -dimensions<sup>1</sup>, we define the superfield  $Q(x^\pm, \theta^\pm)$  as a function of space-time coordinates  $x^\pm$  and anti-commuting coordinates  $\theta^\pm$  and taking values in  $\mathcal{G}/\mathcal{H}$  and it is lifted (locally) to the superfield  $G(x^\pm, \theta^\pm)$  taking values in  $\mathcal{G}$ , with a natural equivalence

$$G_2(x^\pm, \theta^\pm) \sim G_1(x^\pm, \theta^\pm),$$

such that there exists a superfield  $H(x^\pm, \theta^\pm) \in \mathcal{H}$ , such that both are related by a gauge transformation

$$G_2(x^\pm, \theta^\pm) = G_1(x^\pm, \theta^\pm)H(x^\pm, \theta^\pm). \quad (3)$$

For the gauge invariant quantities such as the superfield  $Q(x^\pm, \theta^\pm)$  ordinary derivatives are relevant, while on gauge-covariant quantities such as the superfield  $G(x^\pm, \theta^\pm)$ , the ordinary derivatives have to be replaced by covariant derivatives. We define the gauge-covariant derivative in superspace acting on the superfield  $G(x^\pm, \theta^\pm)$  by

$$\mathcal{D}_\pm G = D_\pm G - iGA_\pm, \quad (4)$$

where  $iGA_\pm \equiv \pi(D_\pm G)$  is the vertical part of  $D_\pm G$ , and  $\mathcal{D}_\pm G \equiv (1-\pi)(D_\pm G)$  is the horizontal part of  $D_\pm G$  [4]. The gauge invariant conserved superfield currents are

$$J_\pm \equiv i\alpha \mathcal{D}_\pm G G^{-1} = i\alpha(1-\pi)D_\pm G G^{-1}, \quad (5)$$

and the gauge-covariant conserved superfield currents

$$K_\pm \equiv -i\alpha G^{-1} \mathcal{D}_\pm G = -i\alpha(1-\pi)G^{-1} D_\pm G, \quad (6)$$

<sup>1</sup> Our notation conventions are as follows. The two dimensional Minkowski metric is  $\eta_{\mu\nu} = \text{diag}(+1, -1)$  and the orthonormal and light-cone coordinates are related by  $x^\pm = \frac{1}{2}(x_0 \pm x_1)$  and  $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$ . Under a Lorentz transformation  $x^\pm$  and  $\partial_\pm$  transform as  $x^\pm \mapsto e^{\mp\Lambda} x^\pm$  and  $\partial_\pm \mapsto e^{\mp\Lambda} \partial_\pm$ , where  $\Lambda$  is the rapidity of the Lorentz boost.

<sup>2</sup> The super derivatives  $D_\pm$  are defined as  $D_\pm = \frac{\partial}{\partial\theta^\pm} - i\theta^\pm \partial_\pm$ ,  $D_\pm^2 = -i\partial_\pm$ ,  $\{D_+, D_-\} = 0$ , where  $\{\cdot, \cdot\}$  is an anti-commutator. The supersymmetry generators are  $Q_\pm = \partial_{\theta^\pm} + i\partial_\pm$ , obeying  $Q_\pm^2 = i\partial_\pm$ .

where  $\alpha$  is some real constant introduced for later convenience. Under the gauge transformation the superfield  $G(x^\pm, \theta^\pm)$  transforms as  $G(x^\pm, \theta^\pm) \rightarrow G(x^\pm, \theta^\pm)H(x^\pm, \theta^\pm)$  and the corresponding gauge invariant and covariant superfield currents transform as

$$J_\pm \rightarrow J_\pm, \quad K_\pm \rightarrow H^{-1}K_\pm H.$$

We define the action of the covariant derivative in superspace on the gauge-covariant superfield currents  $K_\pm$  by

$$\mathcal{D}_\pm K_\mp = D_\pm K_\mp + i\{A_\pm, K_\mp\}.$$

The Lagrangian for the  $\mathcal{G}/\mathcal{H}$  sigma model in superspace is [4]

$$\mathcal{L}_{\mathcal{G}/\mathcal{H}} \equiv \frac{1}{2} \text{Tr} (\mathcal{D}_+ G^{-1} \mathcal{D}_- G) = \frac{1}{2} \text{Tr} (D_+ Q^{-1} D_- Q). \quad (7)$$

The superspace equations of motion obtained from the Lagrangian can be expressed as

$$D_- J_+ - D_+ J_- = 0, \quad (8)$$

$$\mathcal{D}_- K_+ - \mathcal{D}_+ K_- = 0. \quad (9)$$

For the gauge superfields  $A_\pm$ , we can have the following equation for the given homogeneous space  $\mathcal{G}/\mathcal{H}$ :

$$\begin{aligned} D_- A_+ + D_+ A_- &= i\pi \\ &\times \{G^{-1} \mathcal{D}_+ G + iA_+, G^{-1} \mathcal{D}_- G + iA_-\}; \end{aligned} \quad (10)$$

the curvature form in terms of the superfields  $A_\pm$  is

$$\begin{aligned} \mathcal{F}_{-+} &\equiv D_- A_+ + D_+ A_- + i\{A_+, A_-\} \\ &= i\pi \{G^{-1} \mathcal{D}_+ G, G^{-1} \mathcal{D}_- G\}. \end{aligned} \quad (11)$$

In the above expression  $\mathcal{F}_{-+}$  represents the curvature form of the gauge superfields  $A_\pm$ . Our aim in this paper is to develop a superfield Lax formalism of the model in terms of a one-parameter family of flat superfield connections. In the sections which follow, we show that the existence of flat superfield connections is admissible when the target space  $\mathcal{G}/\mathcal{H}$  of the model is a symmetric space. The superfield Lax formalism is then used to derive infinitely many local and non-local superfield conserved quantities.

### 3 Superfield Lax formalism

The superfield Lax formalism of the symmetric space sigma model in superspace can be obtained by defining a one-parameter family of transformations on superfields of the model. A one-parameter family of transformations on the superfields is defined in terms of the matrix superfields  $\mathcal{U}^{(\gamma)}$  which obey the following set of linear differential equations:

$$\begin{aligned} D_+ \mathcal{U}^{(\gamma)} &\equiv i(1-\gamma^{-1})\mathcal{U}^{(\gamma)} J_+ = -\alpha(1-\gamma^{-1})\mathcal{U}^{(\gamma)} \mathcal{D}_+ G G^{-1}, \\ D_- \mathcal{U}^{(\gamma)} &\equiv i(1-\gamma)\mathcal{U}^{(\gamma)} J_- = -\alpha(1-\gamma)\mathcal{U}^{(\gamma)} \mathcal{D}_- G G^{-1}, \end{aligned} \quad (12)$$

where the matrix superfield  $\mathcal{U} \in \mathcal{G}$ . The model retains the zero-curvature representation if the compatibility condition of the linear system (12) becomes equivalent to the following equation:

$$(1 - \gamma^{-1})D_- J_+ + (1 - \gamma)D_+ J_- + i(1 - \gamma^{-1})(1 - \gamma) \{J_+, J_-\} = 0. \tag{13}$$

Using (8) the compatibility condition (13) reduces to the  $\gamma$ -independent equation

$$D_- J_+ + D_+ J_- + 2i \{J_+, J_-\} = 0, \tag{14}$$

which is the zero-curvature condition for the superfield currents  $J_{\pm}$ . If we look at the compatibility condition of the linear system (12), we arrive at the following  $\gamma$ -independent equation:

$$D_- J_+ + D_+ J_- + 2i \{J_+, J_-\} = -i\alpha(2\alpha - 1 - \pi) \times G \{G^{-1}D_+ G, G^{-1}D_- G\} G^{-1}, \tag{15}$$

which does not indicate that the superfield current  $J_{\pm}$  is flat. In order to formulate a theory which gives rise to flat superfield currents, we look at the extra terms appearing on the right-hand side of (15) and look at the constraints which appear in the geometric structure of the target space when these extra terms are set equal to zero. To achieve this we discuss two different cases.

In the first case we assume that  $[\mathbf{k}, \mathbf{k}] \subset \mathbf{h}$ , which implies that

$$(1 - \pi) \{G^{-1}D_+ G, G^{-1}D_- G\} = 0,$$

and (15) reduces to (14) if we choose  $\alpha = 1$ . The matrix superfield  $\mathcal{U}^{(\gamma)}$  generates a one-parameter family of transformations on the solutions of the equations of motion. The superfield  $G(x^{\pm}, \theta^{\pm})$  transforms as

$$G(x^{\pm}, \theta^{\pm}) \rightarrow G^{(\gamma)}(x^{\pm}, \theta^{\pm}) = \mathcal{U}^{(\gamma)} G(x^{\pm}, \theta^{\pm}), \tag{16}$$

where  $G^{(1)}(x^{\pm}, \theta^{\pm}) = G(x^{\pm}, \theta^{\pm})$ . The action of the derivatives  $D_{\pm}$  on the superfield  $G^{(\gamma)}(x^{\pm}, \theta^{\pm})$  will be

$$D_{\pm} G^{(\gamma)} = \gamma^{\mp 1} \mathcal{U}^{(\gamma)} D_{\pm} G + iG^{(\gamma)} \mathcal{A}_{\pm},$$

where  $\mathcal{U}^{(\gamma)} D_{\pm} G$  is the horizontal part and  $iG^{(\gamma)} \mathcal{A}_{\pm}$  is the vertical part of  $D_{\pm} G$ . This implies

$$D_{\pm}^{(\gamma)} G^{(\gamma)} = \gamma^{\mp 1} \mathcal{U}^{(\gamma)} D_{\pm} G, \quad \mathcal{A}_{\pm}^{(\gamma)} = \mathcal{A}_{\pm}, \tag{17}$$

where  $D_{\pm}^{(\gamma)} G^{(\gamma)} = D_{\pm} G^{(\gamma)} - iG^{(\gamma)} \mathcal{A}_{\pm}^{(\gamma)}$ . Thus it can be seen that the Lagrangian (7) is invariant under the transformation (16).

The second case is when  $[\mathbf{k}, \mathbf{k}] \subset \mathbf{k}$ , which is equivalent to saying that

$$\pi \{G^{-1}D_+ G, G^{-1}D_- G\} = 0.$$

Equation (15) reduces to (14), if we choose  $\alpha = \frac{1}{2}$ . For this case, consider the following one-parameter family of differential equations:

$$\begin{aligned} D_+ \mathcal{V}^{(\gamma)} &\equiv i(1 - \gamma^{-1})\mathcal{V}^{(\gamma)} K_+ = \frac{1}{2}(1 - \gamma^{-1})\mathcal{V}^{(\gamma)} G^{-1} D_+ G, \\ D_- \mathcal{V}^{(\gamma)} &\equiv i(1 - \gamma)\mathcal{V}^{(\gamma)} K_- = \frac{1}{2}(1 - \gamma)\mathcal{V}^{(\gamma)} G^{-1} D_- G, \end{aligned} \tag{18}$$

where  $K_{\pm}(x^{\pm}, \theta^{\pm})$  are the components of the gauge-covariant superfields. The  $\mathcal{G}$ -valued matrix superfield  $\mathcal{V}^{(\gamma)}$  transforms under the gauge transformation as

$$\mathcal{V}^{(\gamma)} \rightarrow \mathcal{V}^{(\gamma)} H(x^{\pm}, \theta^{\pm}).$$

By using this gauge transformation, the system (18) can also be expressed as

$$\begin{aligned} D_+ \mathcal{V}^{(\gamma)} &\equiv i(1 - \gamma^{-1})\mathcal{V}^{(\gamma)} K_+ + i\mathcal{V}^{(\gamma)} \mathcal{A}_+ \\ &= \frac{1}{2}(1 - \gamma^{-1})\mathcal{V}^{(\gamma)} G^{-1} D_+ G + i\mathcal{V}^{(\gamma)} \mathcal{A}_+, \\ D_- \mathcal{V}^{(\gamma)} &\equiv i(1 - \gamma)\mathcal{V}^{(\gamma)} K_- + i\mathcal{V}^{(\gamma)} \mathcal{A}_- \\ &= \frac{1}{2}(1 - \gamma)\mathcal{V}^{(\gamma)} G^{-1} D_- G + i\mathcal{V}^{(\gamma)} \mathcal{A}_-. \end{aligned} \tag{19}$$

The compatibility condition of the linear system (19) is

$$\begin{aligned} D_- ((1 - \gamma^{-1}) K_+ + \mathcal{A}_+) + D_+ ((1 - \gamma) K_- + \mathcal{A}_-) \\ + i \{ (1 - \gamma^{-1}) K_+ + \mathcal{A}_+, (1 - \gamma) K_- + \mathcal{A}_- \} \\ = (1 - \gamma^{-1}) D_- K_+ + (1 - \gamma) D_+ K_- \\ + i(1 - \gamma^{-1})(1 - \gamma) \{K_+, K_-\} + \mathcal{F}_{-+}. \end{aligned} \tag{20}$$

For this case, (11) implies that the curvature form  $\mathcal{F}_{-+}$  vanishes so that the compatibility condition (20) reduces to the following  $\gamma$ -independent equation:

$$D_- K_+ + D_+ K_- + 2i \{K_+, K_-\} \equiv -\frac{1}{2} \mathcal{F}_{-+} = 0.$$

The matrix superfields  $\mathcal{U}^{(\gamma)}$  and  $\mathcal{V}^{(\gamma)}$  generate a one-parameter family of transformations on the solutions of the superfield equations which gives rise to a one-parameter family of flat superfield currents. This particular transformation is given by

$$G(x^{\pm}, \theta^{\pm}) \rightarrow G^{(\gamma)}(x^{\pm}, \theta^{\pm}) = \mathcal{U}^{(\gamma)} G(x^{\pm}, \theta^{\pm}) \mathcal{V}^{(\gamma)-1} \mathcal{V}^{(1)}. \tag{21}$$

The action of derivatives  $D_{\pm}$  and  $\mathcal{D}_{\pm}$  on the superfields  $G^{(\gamma)}(x^{\pm}, \theta^{\pm})$  is

$$\begin{aligned} D_{\pm} G^{(\gamma)} &= \gamma^{\mp 1} \mathcal{U}^{(\gamma)} D_{\pm} G \mathcal{V}^{(\gamma)-1} \mathcal{V}^{(1)} + iG^{(\gamma)} \mathcal{A}_{\pm}, \\ \mathcal{D}_{\pm}^{(\gamma)} G^{(\gamma)} &= \gamma^{\mp 1} \mathcal{U}^{(\gamma)} \mathcal{D}_{\pm} G \mathcal{V}^{(\gamma)-1} \mathcal{V}^{(1)}, \quad \mathcal{A}_{\pm} = \mathcal{A}_{\pm}^{(\gamma)}, \end{aligned}$$

where  $\mathcal{D}_{\pm}^{(\gamma)} G^{(\gamma)} = D_{\pm} G^{(\gamma)} - iG^{(\gamma)} \mathcal{A}_{\pm}^{(\gamma)}$ . The Lagrangian (7) is invariant under transformation (21).

In the first case, where we have  $[\mathbf{k}, \mathbf{k}] \subset \mathbf{h}$ , the linear map  $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$  is an isometric Lie algebra automorphism and can be lifted to an isometric Lie group automorphism, which is always true if  $\mathcal{G}$  is simply connected. The coset

space  $\mathcal{G}/\mathcal{H}$  is then a symmetric space [18, 19]. In the second case, where we have  $[\mathbf{k}, \mathbf{k}] \subset \mathbf{k}$ , the coset space  $\mathcal{G}/\mathcal{H}$  is canonically isomorphic to the connected normal Lie subgroup  $\mathcal{K}$  in the Lie group  $\mathcal{G}$  generated by  $\mathbf{k}$  in  $\mathfrak{g}$ , and the action of  $\mathcal{H}$  on  $\mathcal{K}$  by the Lie algebra automorphism  $\sigma$  can be lifted to an action of  $\mathcal{H}$  on  $\mathcal{K}$  by the Lie group automorphism which is always the case if  $\mathcal{K}$  is simply connected. For the special case when  $\mathcal{K}$  is canonically isomorphic to the symmetric space

$$\mathcal{K} \times \mathcal{K} / \Delta \mathcal{K},$$

we get the principal chiral model discussed in Sect. 5. In summary: we have observed that for the first case we have to take  $\mathcal{G}$  to be simply connected and for the second case  $\mathcal{K}$  must be compact. This shows that the supersymmetric sigma model based on  $\mathcal{G}/\mathcal{H}$  admits a Lax formalism and the zero-curvature representation presented in (12) and (15) respectively if  $\mathcal{G}/\mathcal{H}$  is a symmetric space. This is essentially a supersymmetric generalization of earlier work of Eichenerr and Forger [14]. In what follows, we explicitly write the Lax pair of the model and derive an infinite family of superfield conserved quantities.

In both cases, a one-parameter family of flat superfield currents is given by the following transformation rule:

$$\begin{aligned} J_+ &\mapsto J_+^{(\gamma)} = \gamma^{-1} \mathcal{U}^{(\gamma)} J_+ \mathcal{U}^{(\gamma)-1}, \\ J_- &\mapsto J_-^{(\gamma)} = \gamma \mathcal{U}^{(\gamma)} J_- \mathcal{U}^{(\gamma)-1}. \end{aligned}$$

These superfield currents are conserved in superspace for any value of  $\gamma$ :  $D_+ J_-^{(\gamma)} - D_- J_+^{(\gamma)} = 0$ . The associated linear system of the supersymmetric sigma model on the symmetric space can be written as

$$D_{\pm} \mathcal{U}(t, x, \theta; \lambda) = \mathcal{U}(t, x, \theta; \lambda) \mathcal{P}_{\pm}^{(\lambda)}, \quad (22)$$

where the odd superfields  $\mathcal{P}_{\pm}^{(\lambda)}$  are given by

$$\mathcal{P}_{\pm}^{(\lambda)} = \mp \frac{2i\lambda}{(1 \mp \lambda)} J_{\pm}.$$

The parameter  $\lambda$  is the spectral parameter and is related to the parameter  $\gamma$  by  $\lambda = \frac{1-\gamma}{1+\gamma}$ . The compatibility condition of the linear system (22) reduces to a fermionic zero-curvature condition for the odd superfields  $\mathcal{P}_{\pm}^{(\lambda)}$  as follows:

$$\begin{aligned} \{D_+ - \mathcal{P}_+^{(\lambda)}, D_- - \mathcal{P}_-^{(\lambda)}\} &\equiv D_- \mathcal{P}_+^{(\lambda)} + D_+ \mathcal{P}_-^{(\lambda)} \\ &\quad + \{\mathcal{P}_+^{(\lambda)}, \mathcal{P}_-^{(\lambda)}\} = 0. \end{aligned}$$

Now we can define the superspace Grassmann odd operators  $\mathcal{L}_{\pm}^{(\lambda)}$

$$\mathcal{L}_{\pm}^{(\lambda)} = D_{\pm} - \mathcal{P}_{\pm}^{(\lambda)},$$

obeying the (Lax) equations in superspace

$$D_{\mp} \mathcal{L}_{\pm}^{(\lambda)} = \left\{ \mathcal{P}_{\mp}^{(\lambda)}, \mathcal{L}_{\pm}^{(\lambda)} \right\}.$$

By applying  $D_{\pm}$  on (22), one gets a linear system in terms of even superfields  $\tilde{\mathcal{P}}_{\pm}^{(\lambda)}$

$$\partial_{\pm} \mathcal{U}(t, x, \theta; \lambda) = \mathcal{U}(t, x, \theta; \lambda) \tilde{\mathcal{P}}_{\pm}^{(\lambda)}, \quad (23)$$

where the even superfields  $\tilde{\mathcal{P}}_{\pm}^{(\lambda)}$  are given by

$$\tilde{\mathcal{P}}_{\pm}^{(\lambda)} = \left\{ \pm \left( \frac{2\lambda}{1 \mp \lambda} \right) D_{\pm} J_{\pm} - i \left( \frac{2\lambda}{1 \mp \lambda} \right)^2 J_{\pm}^2 \right\}.$$

The compatibility condition of the linear system (23) now reduces to a bosonic zero-curvature condition for the even superfields  $\tilde{\mathcal{P}}_{\pm}^{(\lambda)}$ :

$$\begin{aligned} [\partial_+ - \tilde{\mathcal{P}}_+^{(\lambda)}, \partial_- - \tilde{\mathcal{P}}_-^{(\lambda)}] &\equiv \partial_- \tilde{\mathcal{P}}_+^{(\lambda)} - \partial_+ \tilde{\mathcal{P}}_-^{(\lambda)} \\ &\quad + [\tilde{\mathcal{P}}_+^{(\lambda)}, \tilde{\mathcal{P}}_-^{(\lambda)}] = 0. \end{aligned}$$

The superspace Grassmann even Lax operators  $\tilde{\mathcal{L}}_{\pm}^{(\lambda)}$ ,

$$\tilde{\mathcal{L}}_{\pm}^{(\lambda)} = \partial_{\pm} - \tilde{\mathcal{P}}_{\pm}^{(\lambda)},$$

obey the following equation:

$$\partial_{\mp} \tilde{\mathcal{L}}_{\pm}^{(\lambda)} = [\tilde{\mathcal{P}}_{\mp}^{(\lambda)}, \tilde{\mathcal{L}}_{\pm}^{(\lambda)}].$$

The linear system (23) can be re-expressed in terms of space-time coordinates by

$$\begin{aligned} \partial_0 \mathcal{U}(t, x, \theta; \lambda) &= \mathcal{U}(t, x, \theta; \lambda) \tilde{\mathcal{P}}_0^{(\lambda)}, \\ \partial_1 \mathcal{U}(t, x, \theta; \lambda) &= \mathcal{U}(t, x, \theta; \lambda) \tilde{\mathcal{P}}_1^{(\lambda)}, \end{aligned} \quad (24)$$

with the superfields  $\tilde{\mathcal{P}}_0^{(\lambda)}$  and  $\tilde{\mathcal{P}}_1^{(\lambda)}$  defined by

$$\begin{aligned} \tilde{\mathcal{P}}_0^{(\lambda)} &= \frac{1}{2} \left\{ \left( \frac{2\lambda}{1-\lambda} \right) D_+ J_+ - i \left( \frac{2\lambda}{1-\lambda} \right)^2 J_+^2 \right. \\ &\quad \left. - \left( \frac{2\lambda}{1+\lambda} \right) D_- J_- - i \left( \frac{2\lambda}{1+\lambda} \right)^2 J_-^2 \right\}, \\ \tilde{\mathcal{P}}_1^{(\lambda)} &= \frac{1}{2} \left\{ \left( \frac{2\lambda}{1-\lambda} \right) D_+ J_+ - i \left( \frac{2\lambda}{1-\lambda} \right)^2 J_+^2 \right. \\ &\quad \left. + \left( \frac{2\lambda}{1+\lambda} \right) D_- J_- + i \left( \frac{2\lambda}{1+\lambda} \right)^2 J_-^2 \right\}. \end{aligned}$$

This is the bosonic superfield Lax pair of the model. The compatibility condition of the linear system (24) therefore becomes

$$\begin{aligned} [\partial_0 - \tilde{\mathcal{P}}_0^{(\lambda)}, \partial_1 - \tilde{\mathcal{P}}_1^{(\lambda)}] &\equiv \partial_1 \tilde{\mathcal{P}}_0^{(\lambda)} - \partial_0 \tilde{\mathcal{P}}_1^{(\lambda)} \\ &\quad + [\tilde{\mathcal{P}}_0^{(\lambda)}, \tilde{\mathcal{P}}_1^{(\lambda)}] = 0. \end{aligned}$$

We now define the theory on the spatial interval  $[-a, a]$  and the superfields  $\tilde{\mathcal{P}}_0^{(\lambda)}$  and  $\tilde{\mathcal{P}}_1^{(\lambda)}$  are subjected to the boundary conditions  $\tilde{\mathcal{P}}_0^{(\lambda)}(a) = \tilde{\mathcal{P}}_0^{(\lambda)}(-a)$ ,  $\tilde{\mathcal{P}}_1^{(\lambda)}(a) = \tilde{\mathcal{P}}_1^{(\lambda)}(-a)$ .

The equation satisfied by the superfield monodromy operator  $T_\lambda(x, \theta)$  is

$$\frac{\partial}{\partial x} T_\lambda(x, \theta) = \tilde{\mathcal{P}}_1^{(\lambda)} T_\lambda(x, \theta), \quad (25)$$

with the boundary condition  $T_\lambda(-a) = 1$ . The solution of (25) is

$$T_\lambda(x, \theta) = P \exp \left( - \int_{-a}^x dy \tilde{\mathcal{P}}_1^{(\lambda)}(y, \theta) \right),$$

where  $P$  is the path-ordered operator. The operator  $T_\lambda(x, \theta)$  obeys

$$\frac{\partial}{\partial t} T_\lambda(x, \theta) = \left[ \tilde{\mathcal{P}}_0^{(\lambda)}, T_\lambda(a) \right],$$

which is equivalent to the Lax formalism. This can be used to generate an infinite sequence of local and non-local conservation laws as detailed in the next sections.

## 4 Superfield conserved quantities

### 4.1 Local conserved quantities

We now derive the continuity equations of the superfield local conserved quantities of the model via a set of superfield Riccati equations using the method adopted for the bosonic models (see for example [15, 16]). The equation of motion (9) can be written as

$$\begin{aligned} \mathcal{D}_- K_+ - \mathcal{D}_+ K_- &\equiv \mathcal{D}_- K_+ + i \{ \mathcal{A}_-, K_+ \} \\ &\quad - \mathcal{D}_+ K_- - i \{ \mathcal{A}_+, K_- \} = 0, \end{aligned} \quad (26)$$

where

$$K_\pm \equiv \mathcal{J}_\pm - \mathcal{A}_\pm = -iG^{-1}D_\pm G - \mathcal{A}_\pm.$$

Equation (26) immediately gives

$$2\mathcal{D}_- K_+ + 2i \{ \mathcal{A}_-, K_+ \} = -i \{ K_+, K_- \} - \mathcal{F}_{-+}. \quad (27)$$

Since  $\mathcal{G}/\mathcal{H}$  is a symmetric space, therefore by using (2) the left- and right-hand sides of (27) must vanish separately, i.e.

$$\mathcal{D}_- K_+ = -i \{ \mathcal{A}_-, K_+ \} \text{ in } \mathbf{k}, \quad \mathcal{F}_{-+} = -i \{ K_+, K_- \} \text{ in } \mathbf{h}. \quad (28)$$

These considerations lead to essentially two classes of local conserved quantities; one class consists of currents based on generators of the de Rham cohomology ring of  $\mathcal{G}/\mathcal{H}$ , and the second class consists of currents that are higher-spin generalizations of the super energy momentum tensor [4].

An infinite series of local conservation laws can also be obtained by using a set of compatible Riccati equations. A set of compatible Riccati equations for the symmetric space sigma model in superspace is given by

$$\begin{aligned} D_+ \mathcal{N}(\gamma) &= -i\gamma^{-1} K_+ - i\gamma^{-1} \mathcal{N}(\gamma) K_+ \mathcal{N}(\gamma) \\ &\quad + i[\mathcal{N}(\gamma), \mathcal{A}_+], \\ D_- \mathcal{N}(\gamma) &= i\gamma K_- + i\gamma \mathcal{N}(\gamma) K_- \mathcal{N}(\gamma) + i[\mathcal{N}(\gamma), \mathcal{A}_-], \end{aligned} \quad (29)$$

where  $\mathcal{N} \in \mathcal{G}/\mathcal{H}$  is an even matrix superfield. The following conservation equation immediately follows:

$$\gamma^{-1} D_- \text{Tr}(\mathcal{N}(\gamma) K_+) - \gamma D_+ \text{Tr}(\mathcal{N}(\gamma) K_-) = 0. \quad (30)$$

An infinite sequence of local conservation laws can be obtained by expanding  $\mathcal{N}(\gamma)$  as a power series in  $\gamma$ :  $\mathcal{N}(\gamma) = \sum_{k=0}^{\infty} \gamma^k \mathcal{N}_k$ . On substituting this expansion in (30), one arrives at the algebraic equations obtained successively. The coefficients can be determined by these algebraic equations and substitution of these coefficients yields explicit expressions of the conserved quantities. The details of this depend on the particular model.

The existence of local conserved quantities in supersymmetric models on symmetric spaces can have certain relations with the integrable structures in superstring theory on  $\text{AdS}_5 \times S^5$  where the theory has been regarded as a non-linear sigma model with the field taking values in the supercoset space

$$\frac{PSU(2, 2|4)}{SO(4, 1) \times SO(5)}. \quad (31)$$

The even part of this space is

$$\frac{SO(4, 2)}{SO(4, 1)} \times \frac{SO(6)}{SO(5)} = \text{AdS}_5 \times S^5,$$

which is a symmetric space and therefore admits a Lax formalism (one-parameter family of flat connections) and can further be related to the conserved quantities on the Yang–Mills sector of the AdS/CFT correspondence [5–12]. It is also worthwhile to study how this coincides with the local conserved quantities for the models based on supercoset spaces. In [13] it has been shown that the classical superstring theory on  $\text{AdS}_5 \times S^5$  as a supercoset sigma model admits a Lax formalism which does not imply the existence of local conserved quantities. In the light of our results, we expect that the existence of local conserved quantities might appear from the Lax formalism of the model on the even part of the supercoset space which defines a symmetric space. In all these investigations of integrable structures of classical superstring theory, a target space supersymmetry has been used while the supersymmetric models we have studied involve world-sheet supersymmetry. At this stage we have not been able to relate the integrable structures of these supersymmetric theories. The formalism we have developed can be extended to the model with target space supersymmetry.

### 4.2 Non-local conserved quantities

The non-local conserved quantities have been constructed for both the bosonic as well as the supersymmetric sigma model, via a family of flat currents [20–36]. For our model, we assume spatial boundary conditions such that the superfields  $J_\pm$  vanish as  $x \rightarrow \pm\infty$ . Equation (24), then implies that  $\mathcal{U}(t, \pm\infty, \theta; \lambda)$  are independent of time. The residual freedom in the solution for  $\mathcal{U}(\gamma)$  allows us to fix

$\mathcal{U}(t, -\infty, \theta; \lambda)$  equal to a unit matrix. We are then left with a time independent function,  $\mathcal{Q}(\lambda) = \mathcal{U}(t, \infty; \lambda)$ . Expanding  $\mathcal{Q}(\lambda)$  as a power series in  $\lambda$  gives infinitely many conserved quantities

$$\mathcal{Q}(\lambda) = \sum_{k=0}^{\infty} \lambda^k \mathcal{Q}^{(k)}, \quad \frac{d\mathcal{Q}^{(k)}}{dt} = 0.$$

In order to derive explicit expressions for these conserved quantities in terms of superfields, we write (24) as

$$\begin{aligned} \mathcal{U}(t, x, \theta; \lambda) = & 1 + \frac{1}{2} \int_{-\infty}^x dy \mathcal{U}(t, y, \theta; \lambda) \\ & \times \left\{ \left( \frac{2\lambda}{1-\lambda} \right) D_+ J_+ + \left( \frac{2\lambda}{1+\lambda} \right) D_- J_- \right. \\ & \left. - i \left( \frac{2\lambda}{1-\lambda} \right)^2 J_+^2 + i \left( \frac{2\lambda}{1+\lambda} \right)^2 J_-^2 \right\}. \end{aligned} \tag{32}$$

We expand the superfield  $\mathcal{U}(t, x, \theta; \lambda)$  as a power series in  $\lambda$ ,

$$\mathcal{U}(t, x, \theta; \lambda) = \sum_{k=0}^{\infty} \lambda^k \mathcal{U}_k(t, x, \theta), \tag{33}$$

and compare the coefficients of powers of  $\lambda$ , one gets a series of conserved non-local superfield currents, which upon integration give non-local conserved quantities. The expressions for the first few cases are

$$\begin{aligned} \mathcal{Q}^{(1)a} = & \int_{-\infty}^{\infty} dy (D_+ J_+^a + D_- J_-^a)(t, y, \theta), \\ \mathcal{Q}^{(2)a} = & \int_{-\infty}^{\infty} dy \left( (D_+ J_+^a - D_- J_-^a)(t, y, \theta) \right. \\ & - i f^{abc} (J_+^b J_+^c - J_-^b J_-^c)(t, y, \theta) \\ & + \frac{1}{2} f^{abc} (D_+ J_+^b + D_- J_-^b)(t, y, \theta) \\ & \left. \times \int_{-\infty}^y dz (D_+ J_+^c + D_- J_-^c)(t, z, \theta) \right). \end{aligned}$$

These are the desired non-local conserved quantities which correspond to the bosonic non-local conserved quantities of [14] when the fermions are set to zero. The component content of these superfield conserved quantities is the same as appeared for certain models in [27–33]. The non-local conserved quantities are also known to exist in the classical theory of Green–Schwarz superstrings, where a parameter dependent flat current taking values in the Lie algebra of  $PSU(2, 2|4)$  is shown to exist [5]. The construction of non-local conserved quantities has been extended to the case of the full supercoset space (31) which is not symmetric and the theory also involves a Wess–Zumino term and  $\kappa$ -symmetry [7, 8]. Moreover, it has been shown that Yangian non-local symmetries exist in  $D = 4$  superconformal Yang–Mills theory in the gauge theory sector of the AdS/CFT correspondence [9].

## 5 Examples

### 5.1 Supersymmetric model on complex Grassmannian

In the previous section, we have discussed a general procedure of studying the Lax formalism and extracting conserved quantities for a symmetric space sigma model in superspace. In this section we will discuss an example i.e. the sigma model on the complex Grassmannian manifold  $U(m+n)/U(m) \times U(n)$ . For  $n = 1$ , it reduces to the complex projective space  $CP^m$ . We define a  $U(m+n)$ -valued matrix superfield  $G(x^\pm, \theta^\pm)$

$$G(x^\pm, \theta^\pm) = \begin{pmatrix} X \\ Y \end{pmatrix},$$

$$G^\dagger(x^\pm, \theta^\pm) G(x^\pm, \theta^\pm) = I = G(x^\pm, \theta^\pm) G^\dagger(x^\pm, \theta^\pm),$$

where  $X(x^\pm, \theta^\pm)$  and  $Y(x^\pm, \theta^\pm)$  are superfield matrices of order  $m \times (m+n)$  and  $n \times (m+n)$  respectively. We introduce orthogonal projectors for superfields:

$$P = XX^\dagger, \quad \bar{P} = YY^\dagger, \quad P + \bar{P} = I,$$

which map  $C^{m+n}$  into  $m$  and  $n$  dimensional subspaces spanned by the column vectors of  $X$  and  $Y$  respectively. The super gauge transformation acts on the superfield  $G(x^\pm, \theta^\pm)$  as

$$G(x^\pm, \theta^\pm) = \begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow G'(x^\pm, \theta^\pm) = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

where

$$\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \in U(m) \times U(n).$$

The canonical decomposition of the superfield  $D_\pm G G^{-1}$  is given by

$$\begin{aligned} A_\pm = & \begin{pmatrix} iD_\pm X X^\dagger & 0 \\ 0 & iD_\pm Y Y^\dagger \end{pmatrix}, \\ K_\pm = & \begin{pmatrix} 0 & iD_\pm X Y^\dagger \\ iD_\pm Y X^\dagger & 0 \end{pmatrix}. \end{aligned} \tag{34}$$

The even superfield matrix  $\mathcal{N}$  appearing in (29)–(30), decomposes as

$$\mathcal{N} = \begin{pmatrix} 0 & -\mathcal{M}^\dagger \\ \mathcal{M} & 0 \end{pmatrix}, \tag{35}$$

where  $\mathcal{M}$  is an  $n \times m$  even matrix superfield. The action of the covariant derivative in the superspace on the superfield  $G(x^\pm, \theta^\pm)$  will be

$$\mathcal{D}_\pm \begin{pmatrix} X \\ Y \end{pmatrix} \equiv \begin{pmatrix} \mathcal{D}_\pm X \\ \mathcal{D}_\pm Y \end{pmatrix} = \begin{pmatrix} i\bar{P} \mathcal{D}_\pm X \\ iP \mathcal{D}_\pm Y \end{pmatrix}.$$

As a result of the decompositions, the action for the complex Grassmannian model splits into two parts given by

$$\begin{aligned} \mathcal{L} \equiv & \frac{1}{2} \int d^2x d^2\theta \text{Tr}(\mathcal{D}_+ G^{-1} \mathcal{D}_- G) \\ = & \frac{1}{2} \int d^2x d^2\theta (\text{Tr}(\mathcal{D}_+ X (\mathcal{D}_- X)^\dagger) \\ & + \text{Tr}(\mathcal{D}_+ Y (\mathcal{D}_- Y)^\dagger)). \end{aligned}$$

The variation of the action yields the following equations of motion for each of the superfields  $X$  and  $Y$ :

$$\begin{aligned} \mathcal{D}_+ \mathcal{D}_- X - X(\mathcal{D}_+ X)^\dagger \mathcal{D}_- X &= 0, \\ \mathcal{D}_+ \mathcal{D}_- Y - Y(\mathcal{D}_+ Y)^\dagger \mathcal{D}_- Y &= 0. \end{aligned} \tag{36}$$

We rewrite the above equations of motion in terms of projector superfields:

$$[D_+ D_- P, P] = 0 = [D_+ D_- \bar{P}, \bar{P}].$$

The one-parameter family of transformations acts on the superfields  $X$  and  $Y$  as

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \mathcal{U}^{(\gamma)} \begin{pmatrix} X \\ Y \end{pmatrix},$$

and the corresponding linear system can be expressed as

$$\begin{aligned} D_+ \mathcal{U}^{(\gamma)} &\equiv -i(1 - \gamma^{-1}) \mathcal{U}^{(\gamma)} \mathcal{D}_+ \begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}^\dagger, \\ D_- \mathcal{U}^{(\gamma)} &\equiv -i(1 - \gamma) \mathcal{U}^{(\gamma)} \mathcal{D}_- \begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}^\dagger, \end{aligned}$$

where

$$-\mathcal{D}_\pm \begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}^\dagger = [D_\pm P, P] = [D_\pm \bar{P}, \bar{P}].$$

The projector superfields  $P$  and  $\bar{P}$  transform according to the law

$$P \rightarrow \mathcal{U}^{(\gamma)} P \mathcal{U}^{(\gamma)\dagger}, \quad \bar{P} \rightarrow \mathcal{U}^{(\gamma)} \bar{P} \mathcal{U}^{(\gamma)\dagger}.$$

One can apply the same procedure to the real Grassmannian manifold. These considerations are sufficient for the construction of the conserved quantities of the model. The manifold is a symmetric space when we define an involutive automorphism  $\sigma$  acting on the superfield  $G$  as

$$\sigma(G) = \Theta G \Theta^{-1},$$

where

$$G(x^\pm, \theta^\pm) \in U(m+n), \quad \Theta = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}.$$

Using (34) and (35), (29) and (30) generate an infinite series of local conservation laws in terms of the superfields  $B_\pm = iD_\pm X X^\dagger$ ,  $C_\pm = iD_\pm Y Y^\dagger$  and  $K_\pm$ . The set of compatible Riccati differential equations for an even matrix superfield  $\mathcal{M}$  is

$$\begin{aligned} D_+ \mathcal{M}(\gamma) &= -i\gamma^{-1} K_+ + i\gamma^{-1} \mathcal{M}(\gamma) K_+^\dagger \mathcal{M}(\gamma) \\ &\quad - iC_+ \mathcal{M}(\gamma) + i\mathcal{M}(\gamma) B_+, \\ D_- \mathcal{M}(\gamma) &= i\gamma K_- - i\gamma \mathcal{M}(\gamma) K_-^\dagger \mathcal{M}(\gamma) - iC_- \mathcal{M}(\gamma) \\ &\quad + i\mathcal{M}(\gamma) B_-. \end{aligned}$$

This set can immediately be used to derive an infinite sequence of conservation laws:

$$\begin{aligned} \gamma^{-1} D_- \text{Tr} \left( \mathcal{M}^\dagger(\gamma) K_+ + \mathcal{M}(\gamma) K_+^\dagger \right) \\ - \gamma D_+ \text{Tr} \left( \mathcal{M}^\dagger(\gamma) K_- + \mathcal{M}(\gamma) K_-^\dagger \right) = 0. \end{aligned}$$

Expanding  $\mathcal{M}(\gamma)$  as a power series in  $\gamma$ :  $\mathcal{M}(\gamma) = \sum_{k=0}^\infty \gamma^k \mathcal{M}_k$ , one can generate  $\gamma$ -independent conservation laws.

### 5.2 Supersymmetric principal chiral model (SPCM)

In this section, we discuss the supersymmetric principal chiral model (SPCM) as a symmetric space model. If we suppose that  $\mathcal{H}$  is a trivial subgroup of  $\mathcal{G}$ , setting  $\mathcal{A}_\pm = 0$ ,  $K_\pm$  becomes

$$K_\pm \rightarrow -iG^{-1} D_\pm G = \mathcal{J}_\pm.$$

Equation (27) becomes

$$D_- \mathcal{J}_+ = -\frac{i}{2} \{ \mathcal{J}_+, \mathcal{J}_- \}. \tag{37}$$

The Lie group  $\mathcal{G}$  can now be considered as a symmetric space: let

$$\Delta\mathcal{G} = \{ (G, G) \mid G \in \mathcal{G} \},$$

be the diagonal of  $\mathcal{G} \times \mathcal{G}$ , and define  $\sigma : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$  such that

$$\sigma(G, G') = (G', G).$$

Then  $\mathcal{G} \times \mathcal{G} / \Delta\mathcal{G}$  is a symmetric space with involution  $\sigma$ . We define a map  $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$  such that the pair  $(G_1, G_2)$  is mapped to  $G = G_1 G_2^{-1}$ . In this case the decomposition of the corresponding Lie algebra will be

$$\mathfrak{g} + \mathfrak{g} = \mathfrak{h} + \mathfrak{k}.$$

By writing the superfield  $G = (G_1, G_2)$  taking values in  $\mathcal{G} \times \mathcal{G}$ , the gauge superfield can be expressed as

$$\begin{aligned} \mathcal{A}_\pm = \left( \left( -\frac{i}{2} G_1^{-1} D_\pm G_1 - \frac{i}{2} G_2^{-1} D_\pm G_2 \right), \right. \\ \left. \left( -\frac{i}{2} G_1^{-1} D_\pm G_1 - \frac{i}{2} G_2^{-1} D_\pm G_2 \right) \right). \end{aligned}$$

The one-parameter family of transformations on these fields is given by

$$\begin{aligned} (G_1, G_2) \rightarrow (G_1^{(\gamma)}, G_2^{(\gamma)}) &= (\mathcal{U}_1^{(\gamma)} G_1, \mathcal{U}_2^{(\gamma)} G_2) \\ &= (\mathcal{U}_1^{(\gamma)}, \mathcal{U}_2^{(\gamma)}) (G_1, G_2), \end{aligned}$$

where  $\mathcal{U}_1^{(\gamma)}$  and  $\mathcal{U}_2^{(\gamma)}$  belong to  $\mathcal{G}$ . The transformation of the superfield  $G(x^\pm, \theta^\pm)$  is therefore

$$G(x^\pm, \theta^\pm) \mapsto G^{(\gamma)}(x^\pm, \theta^\pm) = \mathcal{U}_1^{(\gamma)} G(x^\pm, \theta^\pm) \mathcal{U}_2^{(\gamma)-1}. \tag{38}$$

Here we choose the boundary values  $\mathcal{U}_1^{(1)} = 1$ ,  $\mathcal{U}_2^{(1)} = 1$  or  $G^{(1)} = G$ . The set of linear differential equations satisfied by  $\mathcal{U}_1^{(\gamma)}$  and  $\mathcal{U}_2^{(\gamma)}$  are

$$\begin{aligned} (D_+\mathcal{U}_1^{(\gamma)}, D_+\mathcal{U}_2^{(\gamma)}) &= -(1-\gamma^{-1}) (\mathcal{U}_1^{(\gamma)}, \mathcal{U}_2^{(\gamma)}) \\ &\quad \times \mathcal{D}_+(G_1, G_2)(G_1, G_2)^{-1}, \\ (D_-\mathcal{U}_1^{(\gamma)}, D_-\mathcal{U}_2^{(\gamma)}) &= -(1-\gamma) (\mathcal{U}_1^{(\gamma)}, \mathcal{U}_2^{(\gamma)}) \\ &\quad \times \mathcal{D}_-(G_1, G_2)(G_1, G_2)^{-1}. \end{aligned} \quad (39)$$

Evaluating the covariant derivative and using  $G = G_1 G_2^{-1}$ , we arrive at

$$\begin{aligned} (D_+\mathcal{U}_1^{(\gamma)}, D_+\mathcal{U}_2^{(\gamma)}) &= i(1-\gamma^{-1}) \\ &\quad \times (i\mathcal{U}_1^{(\gamma)} D_+ G G^{-1}, -i\mathcal{U}_2^{(\gamma)} G^{-1} D_+ G), \\ (D_-\mathcal{U}_1^{(\gamma)}, D_-\mathcal{U}_2^{(\gamma)}) &= i(1-\gamma) \\ &\quad \times (i\mathcal{U}_1^{(\gamma)} D_- G G^{-1}, -i\mathcal{U}_2^{(\gamma)} G^{-1} D_- G). \end{aligned}$$

If we take  $\mathcal{U}_1^{(\gamma)} = \mathcal{U}^{(\gamma)}$  and  $\mathcal{U}_2^{(\gamma)} = \mathcal{V}^{(\gamma)}$ , (38) and (39) reduce to the following equations:

$$G(x^\pm, \theta^\pm) \mapsto G^{(\gamma)}(x^\pm, \theta^\pm) = \mathcal{U}^{(\gamma)} G(x^\pm, \theta^\pm) \mathcal{V}^{(\gamma)-1}, \quad (40)$$

$$D_+\mathcal{U}^{(\gamma)} = \frac{i}{2}(1-\gamma^{-1})\mathcal{U}^{(\gamma)} \mathcal{J}_+^L, \quad (41)$$

$$D_-\mathcal{U}^{(\gamma)} = \frac{i}{2}(1-\gamma)\mathcal{U}^{(\gamma)} \mathcal{J}_-^L, \quad (42)$$

$$D_+\mathcal{V}^{(\gamma)} = \frac{i}{2}(1-\gamma^{-1})\mathcal{V}^{(\gamma)} \mathcal{J}_+^R, \quad (43)$$

$$D_-\mathcal{V}^{(\gamma)} = \frac{i}{2}(1-\gamma)\mathcal{V}^{(\gamma)} \mathcal{J}_-^R, \quad (44)$$

where  $\mathcal{J}_\pm^L = iD_\pm G G^{-1}$  and  $\mathcal{J}_\pm^R = -iG^{-1}D_\pm G$ . The compatibility conditions for these equations are obtained by applying  $D_-$  to (41) and (43) and  $D_+$  to (42) and (44), so that one gets

$$\begin{aligned} \mathcal{U}^{(\gamma)} \left\{ (1-\gamma^{-1})D_-\mathcal{J}_+^L + (1-\gamma)D_+\mathcal{J}_-^L \right. \\ \left. + i(1-\frac{1}{2}(\gamma+\gamma^{-1}))\{\mathcal{J}_+^L, \mathcal{J}_-^L\} \right\} = 0, \\ \mathcal{V}^{(\gamma)} \left\{ (1-\gamma^{-1})D_-\mathcal{J}_+^R + (1-\gamma)D_+\mathcal{J}_-^R \right. \\ \left. + i(1-\frac{1}{2}(\gamma+\gamma^{-1}))\{\mathcal{J}_+^R, \mathcal{J}_-^R\} \right\} = 0. \end{aligned}$$

We see that the supersymmetric principal chiral model is in fact an integrable supersymmetric sigma model on a symmetric space and can easily be extracted from a supersymmetric sigma model on a general symmetric space.

Let us write an arbitrary element of  $\mathcal{G} \times \mathcal{G}$  in the form

$$\Gamma = \begin{pmatrix} G_L & 0 \\ 0 & G_R \end{pmatrix}$$

and consider the symmetric space sigma model that corresponds to the involutive automorphism given by

$$\Sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can now construct the superfields

$$\tilde{\Gamma} \equiv \Sigma G^{-1} \Sigma = \begin{pmatrix} G_R^{-1} & 0 \\ 0 & G_L^{-1} \end{pmatrix},$$

and

$$M \equiv \tilde{\Gamma} \Gamma = \begin{pmatrix} G_R^{-1} G_L & 0 \\ 0 & G_L^{-1} G_R \end{pmatrix}.$$

The subgroup consists of diagonal elements for which  $G_R = G_L$ , and we write  $G = G_R^{-1} G_L$ , so that

$$M = \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix}.$$

Then the superfield conserved currents of the model can be written as

$$\mathcal{J}_\pm \equiv iM^{-1}D_\pm M = \begin{pmatrix} iG^{-1}D_\pm G & 0 \\ 0 & iGD_\pm^{-1}G \end{pmatrix}.$$

The construction then yields the superfield Lax formalism of the supersymmetric principal chiral model (SPCM). The Lax formalism of SPCM is responsible for the existence of an infinite sequence of local and non-local conserved quantities [36].

The superspace equation of motion (37) implies an infinite series of local conservation laws [35, 36]:

$$D_\pm \text{Tr} (\mathcal{J}_\mp)^m = 0, \quad D_\pm \text{Tr} (\mathcal{J}_\mp^{m-1} \mathcal{J}_{\mp\mp}) = 0,$$

with  $\mathcal{J}_{\mp\mp} = D_\mp \mathcal{J}_\mp + i\mathcal{J}_\mp^2$ , where the values of  $m$  are precisely the exponents of the Lie algebra of  $\mathcal{G}$ . The local conserved quantities of SPCM also arise from the Lax formalism via a super Bäcklund transformation (SBT) or equivalently from super Riccati equations [36]. One can easily obtain an infinite sequence of non-local conserved quantities for the SPCM. The expressions for the first two non-local conserved quantities are

$$\begin{aligned} \tilde{Q}^{(1)a} &= \frac{1}{2} \int_{-\infty}^{\infty} dy (D_+\mathcal{J}_+^a + D_-\mathcal{J}_-^a)(t, y, \theta), \\ \tilde{Q}^{(2)a} &= \int_{-\infty}^{\infty} dy \left( \frac{1}{2} (D_+\mathcal{J}_+^a - D_-\mathcal{J}_-^a)(t, y, \theta) \right. \\ &\quad - \frac{i}{4} f^{abc} (\mathcal{J}_+^b \mathcal{J}_+^c - \mathcal{J}_-^b \mathcal{J}_-^c)(t, y, \theta) \\ &\quad + \frac{1}{8} f^{abc} (D_+\mathcal{J}_+^b + D_-\mathcal{J}_-^b)(t, y, \theta) \\ &\quad \left. \times \int_{-\infty}^y dz (D_+\mathcal{J}_+^c + D_-\mathcal{J}_-^c)(t, z, \theta) \right). \end{aligned}$$

These conserved quantities are exactly the same as obtained in [36]. The component contents gives bosonic conserved quantities which generates a Yangian with two copies corresponding to left and right currents [33–36].



## 6 Concluding remarks

We have investigated a one-parameter family of flat superfield connections of the supersymmetric sigma model based on symmetric spaces. This suggests that the model in superspace represents an integrable system exhibiting the Lax formalism and the existence of an infinite number of local and non-local conserved quantities. Some explicit examples are given to illustrate the results. The work can be extended into a number of directions. The study that needs a consideration immediately is the quantization of the conserved quantities which could eventually lead to the implications of the  $S$ -matrices of these models. It will be interesting to develop a similar formalism for the superspace sigma models based on supercoset spaces which appear in the superstring theory on the  $AdS_5 \times S^5$  background and its relation to Yangian symmetry. The  $r$ -matrix formalism of the superspace sigma models based on bosonic symmetric spaces and supercoset spaces is also a direction which needs to be investigated. Moreover many of the integrability structures which have appeared in the AdS/CFT correspondence can be further extended to incorporate certain mathematical techniques of integrable field theories such as involution of local conserved quantities, their quantization,  $S$ -matrix and  $r$ -matrix formalism, algebraic and thermodynamics Bethe Ansatz etc.

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